Abstract

The finite element method is the matter of unflagging interest since the late fifties of the previous century. Most often it is based on compatible displacement field and uses the principle of minimum of potential energy. Despite of numerous applications of kinematically admissible approach new models are created so as to avoid its inadequacies and to attain other desirable features of the solution.
An interesting alternative to the existing displacement-based models may be the stress-based approach which employs statically admissible quantities as degrees of freedom. It is rarely used by scientists and engineers because of some difficulties that emerge when a self balanced stress field is constructed. It seems to be more complicated than using a compatible displacement field. However, it is still desirable to search for such solutions as their advantages overbalance initial difficulties.
The aim of this work is to present a complete equilibrium approach to FEM in the problem of thin plate bending. Innumerable formulations of the finite elements exist in this field of structure analysis, but they are usually displacement-based, mixed or hybrid ones. A different formulation is considered in the course of the present work. It makes use of stress functions and it is created on basis of the principle of minimum of complementary energy. The equilibrium equations are satisfied identically by employment of Southwell's stress functions [136] by which bending and twisting moments are described. This concept was mentioned by Fraeijs de Veubeke and Zienkiewicz [48]. The first attempts of employment of the equilibrium model were made by Elias [41] and Morley [110].
The main part o this work comprises the stress-based approach to FEM in analysis of the plate bending problem. The Southwell stress functions have been approximated by use of linear or quadratic shape functions. Two types of triangular finite elements (3-node and 6-node triangles) have been used. Boundary conditions, which in the case of presented method take form of linear combinations of degrees of freedom (Southwell's stress functions), have been satisfied by means of Lagrange multiplier method. Two types of employment of Lagrange multipliers have been proposed. Firstly, they are engaged as degrees of freedom of boundary elements (*multi point constraints elements*) so as to impose values of normal bending moment and effective shearing force along the edge of the plate. Secondly, they are introduced at plate element vertices (as additional degree of freedom) in order to assure equilibrium of lateral forces applied to the node. Additionally, another possibility has been proposed to fulfil conditions of the free edge of the plate. It is done with use of penalty function method.

The transverse displacement field has been found by means of the least squares method. The grounds for this concept are curvatures at integration points calculated on basis of constitutive relations in the equilibrium approach to finite element method. The plate is then discretised with use of Bogner-Fox-Schmit elements [17]. Curvatures at the same integration points are expressed in terms of unknown degrees of freedom of the BFS element. Differences between these two formulations of curvatures are minimized by the least squares method. This leads to finding BFS element parameters and consequently, to designation of approximate displacement field (on basis of BFS shape functions).
A part of this study embraces also an *a posteriori* evaluation of the relative error of approximate solution. It is done through the method by Prager and Synge [125], [140]. It uses kinematically and statically admissible solutions which are the output of the displacement-based and stress-based methods, respectively. The lower and upper bounds for the relative error of approximate solution may be found and the calculated values are completely satisfactory.
After a detailed description of the methodology presented in course of this research there is an overview of numerical results that comprises analysis of statics of elastic (isotropic and orthotropic) plates. The obtained results are compared with the ones received by means of displacement-based method (Hsieh-Cluogh-Tocher 3-node and 6-node triangles [28]). A similar convergence rate has been observed which clearly shows that the presented method is an attractive alternative to the existing and widely used approaches. Among most valuable advantages of the stress-based formulation we may distinguish:

- a low order of interpolation polynomials that guarantees convergence (linear functions), which is impossible in the case of displacement-based method where quadratic polynomials are necessary if incompatible elements are concerned and cubic functions should be employed when conforming elements are considered,
- a possibility to find the lower bound to the ultimate loading, which improves design safety,
- a possibility of calculation of lower and upper bounds to the strain energy when implementation of equilibrium and displacement FEM is made,
- easy estimation of the error of approximate solution provided that statically and kinematically admissible solutions are at our disposal.

Finally, the duality of available solutions should not be underestimated. Both methods, the displacement-based and the stress-based ones, are characterized by monotonic convergence to the exact solution and offer us lower and upper bounds for the strain energy. This guarantees that both the methods approach the exact solution if the gap between values of the energy decreases within mesh refinement.